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TRANSVERSE VIBRATIONS OF A RECTANGULAR MEMBRANE WITH DISCONTINUOUSLY VARYING DENSITY

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1. INTRODUCTION

The dynamic behavior of structural systems with continuously or discontinuously varying material properties is of considerable interest in several areas of applied science and technology [1–3]. The present note deals with the study of a vibrating rectangular membrane with a discontinuously varying density distribution (Figure 1).

The problem is solved by expanding the displacement amplitude in terms of a double Fourier Series which constitutes the exact solution in the case of an homogeneous membrane. The Rayleigh–Ritz method is then used to generate the determinantal equation.

2. APPROXIMATE ANALYTICAL SOLUTION

In the case of normal modes the problem is governed by the differential system

$$\nabla^2 W + \omega^2 \frac{\rho}{T} W = 0, \quad W(\partial D_2) = 0, \qquad (1a, b)$$

with

$$\rho(\bar{x}, \bar{y}) = \begin{cases} \rho_1 \ \forall \ (\bar{x}, \bar{y}) \in D_1, \\ \rho_2 \ \forall \ (\bar{x}, \bar{y}) \in D_2 - D_1 \end{cases}. \qquad (2)$$

From the point of view of finding an approximate solution it is convenient to formulate the problem in terms of the functional

$$J[W = \int\int_{D_2} (W_{\bar{x}}^2 + W_{\bar{y}}^2)\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{y} - \frac{\omega^2}{T} \rho_2 \left(\gamma \int\int_{D_1} W^2\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{y} + \int\int_{D_2 - D_1} W^2\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{y} \right), \qquad (2)$$

where

$$\gamma = \rho_1/\rho_2, \quad D_1 \subset D_2.$$

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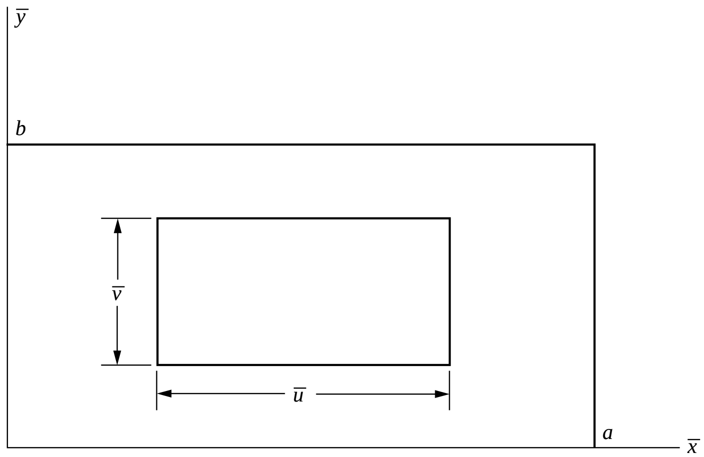

Figure 1. Non-homogeneous rectangular membrane considered in the present study $(u = \bar{u}/a = \nu = \bar{\nu}/b)$.

Introducing the dimensionless variables $x = \bar{x}/a, y = \bar{y}/b$, expression (3) becomes

$$\lambda J[W] = \int\int_{C_2} (W_x^2 + \lambda^2 W_y^2)\, \mathrm{d}x\, \mathrm{d}y - \frac{\omega^2 \rho_2 a^2}{T} \left(\gamma \int\int_{C_1} W^2\, \mathrm{d}x\, \mathrm{d}y + \int\int_{C_2 - C_1} W^2\, \mathrm{d}x\, \mathrm{d}y\right), \quad (4)$$

where $\lambda = a/b$.

The displacement amplitude will now be approximated by a truncated Fourier series of the form

$$W \cong W_a = \sum_{n=1}^{N} \sum_{m=1}^{M} b_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}. \quad (5)$$

Substituting equation (5) in equation (4) and requiring

$$\frac{\partial J}{\partial b_{nm}}[W_a] = 0, \quad (6)$$

one obtains a linear system of equations in the b_{nm}'s. The non-trivial solution of the system leads, finally, to the determinantal equation in the eigenvalues of the problem.

TABLE 1
Values of the fundamental frequency coefficient $\Omega_1 = \sqrt{\rho_2/T}\,\omega_1 a$ *as a function of* λ, γ *and* $u = v$ (*Figure* 1)

γ	u	$\lambda = 1$			$\lambda = 1{\cdot}5$			$\lambda = 2$		
		W_1	W_2	W_8	W_1	W_2	W_8	W_1	W_2	W_8
0·1		4·776	4·765	4·75667	6·089	6·078	6·06862	7·552	7·541	7·52943
3	0·2	3·897	3·877	3·85974	4·967	4·95	4·92989	6·161	6·142	6·11907
10		2·899	2·819	2·74548	3·695	3·619	3·53542	4·583	4·501	4·40164
0·1		5·235	5·182	5·16617	6·673	6·624	6·60432	8·277	8·224	8·19986
3	0·3	3·488	3·456	3·44392	4·447	4·417	4·40214	5·516	5·483	5·46554
10		2·279	2·211	2·18316	2·906	2·839	2·80628	3·604	3·532	3·49159
0·1		5·96	5·809	5·78831	7·598	7·458	7·43036	9·424	9·274	9·23879
3	0·4	3·151	3·123	3·11758	4·017	3·99	3·98372	4·982	4·953	4·9455
10		1·904	1·862	1·85357	2·427	2·387	2·37616	3·01	2·966	2·95344

3. NUMERICAL RESULTS

The present investigation deals with the determination of the lower natural frequencies of symmetric normal modes of the system depicted in Figure 1. For this configuration $\bar{u} = \bar{u}/a = v = \bar{v}/b$ the inner, concentric portion possesses the same aspect ratio as the outer boundary of the membrane.

The frequency coefficients were determined using the terms: $n = 1$, $m = 1$, 3, 5, 7; $n = 3$, $m = 3$, 5, 7; $n = 5$, $m = 5$.

Table 1 depicts values of the fundamental frequency coefficient $\Omega_1 = \sqrt{\rho_2/T}\,\omega_1 a$ for several combinations of values of λ, γ and $u = v$ when one, two and eight terms of the approximating function are employed. Table 2 shows the second frequency coefficient corresponding to a symmetric mode when one and eight terms of the truncated double Fourier series are used. From the analysis of

TABLE 2
Values of the frequency coefficient $\Omega_2 = \sqrt{\rho_2/T}\,\omega_2\, a$ *in the case of symmetric modes* (*Figure* 1)

γ	u	$\lambda = 1$		$\lambda = 1{\cdot}5$		$\lambda = 2$	
		W_2	W_8	W_2	W_8	W_2	W_8
0·1		10·62	10·4668	15·471	15·1337	20·411	19·7972
3	0·2	9·19	9·03024	13·379	13·0143	17·645	17·0086
10		8·464	8·05261	12·252	11·3749	16·126	14·7275
0·1		11·403	10·9616	16·576	15·5947	21·851	20·1661
3	0·3	8·993	8·80481	13·078	12·6817	17·242	16·5803
10		8·198	7·15777	11·861	10·1834	15·608	13·2428
0·1		12·422	11·579	17·979	16·1315	23·664	20·6305
3	0·4	8·729	8·37533	12·696	12·0911	16·739	15·8351
10		7·216	5·54573	10·462	7·9857	13·777	10·4495

TABLE 3
Comparison of values of Ω_1 in the case of a square non-homogeneous membrane

γ	$u = v$	Ω_1 Present study	Ω_1 Reference [1]
0·1		5·166	5·17
3	0·3	3·444	3·54
10		2·183	2·34

Tables 1 and 2 one concludes that the rate of convergence appears to be satisfactory.

Finally, Table 3 depicts a comparison of fundamental frequency coefficients in the case of a square, non-homogeneous membrane between the results obtained in the present investigation and those determined in reference [1], where a single polynomial expression was used to represent, the fundamental mode shape. The agreement is excellent in the case of moderate values of γ leading to the conclusion that a simple, polynomial co-ordinate function yields very good accuracy in the case of a rather complex elastodynamics problem.

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